

EXHAUSTING FAMILIES OF REPRESENTATIONS AND SPECTRA OF PSEUDODIFFERENTIAL OPERATORS

VICTOR NISTOR AND NICOLAS PRUDHON

ABSTRACT. Families of representations of suitable Banach algebras provide a powerful tool in the study of the spectral theory of (pseudo)differential operators and of their Fredholmness. We introduce the new concept of an exhausting family of representations of a C^* -algebra A . An *exhausting family* of representations of a C^* -algebra A is a set \mathcal{F} of representations of A with the property that every irreducible representation of A is weakly contained in some $\phi \in \mathcal{F}$. An exhausting family \mathcal{F} of representations of A has the property that “ $a \in A$ is invertible if, and if, $\phi(a)$ is invertible for any $\phi \in \mathcal{F}$.” Consequently, the spectrum of a is given by $\text{Spec}(a) = \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$. In other words, every exhausting family of representations is invertibility sufficient, a concept introduced by Roch in *Algebras of approximation sequences: structure of fractal algebras* (2003). We prove several properties of exhausting families and we provide necessary and sufficient conditions for a family of representations to be exhausting. Using results of Ionescu and Williams (Indiana Univ. Math. J. 2009), we show that the regular representations of amenable, second countable, locally compact groupoids with a Haar system form an exhausting family of representations. If A is a separable C^* -algebra, we show that a family \mathcal{F} of representations of A is exhausting if, and only if, it is invertibility sufficient. However, this result is not true, in general, for non-separable C^* -algebras. With an eye towards applications, we extend our results to the case of unbounded operators. A typical application of our results is to parametric families of differential operators arising in the analysis on manifolds with corners, in which case we recover the fact that a parametric operator P is invertible if, and only if, its Mellin transform $\hat{P}(\tau)$ is invertible, for all $\tau \in \mathbb{R}^n$. In view of possible applications, we have tried to make this paper accessible to non-specialists in C^* -algebras.

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INTRODUCTION

A typical result in spectral theory of N -body Hamiltonians [9, 18, 19, 23, 27] associates to the Hamiltonian H a family of other operators H_ϕ , $\phi \in \mathcal{F}$, such that the essential spectrum $\text{Spec}_{ess}(H)$ of H is obtained in terms of the usual spectra $\text{Spec}(H_\phi)$ of H_ϕ as the closure of the union of the later:

$$(1) \quad \text{Spec}_{ess}(H) = \overline{\bigcup_{\phi \in \mathcal{F}} \text{Spec}(H_\phi)}.$$

It was noticed that sometimes the closure is not necessary, and one of the motivations of our paper is to clarify this issue. Our approach is based on the well known fact that the operators H_ϕ are obtained as homomorphic images (in a suitable sense) of the operator H , that is $H_\phi = \phi(H)$, where the morphisms ϕ are part of a suitable family of representations \mathcal{F} of a certain C^* -algebra associated to H . This justifies the study of families of representations. See for example [19] for an illustration of this approach. As a note on our terminology, by *morphism* and *representation* of C^* -algebras, we shall always mean a **-morphism*, respectively, a **-representation*.

Another, related, motivation comes from the characterization of Fredholm integral operators [9, 31, 37, 42, 43, 44, 45]. We are especially interested in the approach to this question using groupoids [11, 12, 24, 25, 46]. More precisely, for suitable manifolds M and for differential operators D on M compatible with the geometry, there was devised a procedure to associate to M the following data: (i) spaces Z_α , $\alpha \in I$; (ii) groups G_α , $\alpha \in I$; and (iii) G_α -invariant differential operators D_α acting on $Z_\alpha \times G_\alpha$. This data can be used to characterize the Fredholm property of D as follows. Let m be the order of D , then

$$(2) \quad D : H^s(M) \rightarrow H^{s-m}(M) \text{ is Fredholm} \Leftrightarrow D \text{ is elliptic and} \\ D_\alpha \text{ is invertible for all } \alpha \in I.$$

Moreover, the spaces Z_α and the groups G_α are independent of D . If M is compact (without boundary), then the index I is empty (so there are no D_α s). In general, for non-compact manifolds, the conditions on the operators D_α are, nevertheless, necessary. The non-compact geometries to which this characterization of Fredholm operators applies include: asymptotically euclidean manifolds, asymptotically hyperbolic manifolds, manifolds with poly-cylindrical ends, and many others (see [32, 33] for surveys). Again, the operators D_α are homomorphic images of the operator D , which leads us again to the study of families of representations.

The results in [18, 19, 24] mentioned above are the main motivation for this work, which is a purely theoretical one on the representation theory of C^* -algebras, even though the applications are to spectral theory and (pseudo)differential operators.

Our main results concern “exhausting families of representations,” a concept that we introduce and study in this paper. To explain our results, let us discuss first the important, related concept of an “invertibility sufficient family of representations.”

Recall [41] that an *invertibility sufficient family* of representations \mathcal{F} of a unital C^* -algebra A is a set of representations with the property that $a \in A$ is invertible if, and only if, $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$. This concept is directly applicable to the problems mentioned in the beginning of this introduction. It is equivalent to the concept of a *strictly norming family* of representations [17, 41], a concept that we recall in the main body of the paper. In practice, it is not straightforward to check that a family of representations is invertibility sufficient or strictly norming. Motivated by this, we introduce an *exhausting family* of representations of A as a set \mathcal{F} with the property that every irreducible representation of A is weakly contained in a representation $\phi \in \mathcal{F}$. Exhausting families of representations turn out to have many useful properties.

Here are the contents of the sections of the paper and our main results. In the following section—the second section—we discuss some results on faithful family of representations in preparation and as motivation for the study of exhausting families of representations, which is the main thrust of the third section. Thus, in the third section, we discuss and prove various basic properties of exhausting families. We also discuss their relation with invertibility sufficient families of representations. We prove that the C^* -algebras of groupoids \mathcal{G} that satisfy the Effros-Hahn conjecture and have amenable isotropy groups have the property that the family of regular representations $\mathcal{R} = \{\pi_x\}$ is exhausting (here x is ranging through the units of \mathcal{G}). We notice that an example due to Voiculescu shows that this result is not true in general. In the fourth section we provide a necessary and sufficient conditions for a family of representations of A to be exhausting in terms of the topology on the primitive ideal spectrum $\text{Prim}(A)$ of A . In particular, we show that for a separable C^* -algebra, a set of representations of A is invertibility sufficient if, and only if, it is exhausting. We also provide an example of an invertibility sufficient family that is not exhausting in the non-separable case. The fifth section contains some material that allows us to treat also unbounded operators affiliated to a C^* -algebra. The last section—the sixth—contains a typical application of our results to parametric families of differential operators. This type of operators arises in the analysis on manifolds with corners (more precisely, in the case of manifolds with poly-cylindrical ends). In that case, we recover the fact that an operator compatible with the geometry is invertible if, and only if, its Mellin transform is invertible. Due to the fact that the main applications are to areas other than the study of C^* -algebras, we have written the paper with an eye towards the non-specialist in C^* -algebras. In particular, in addition to the relevant references, we have also included a few short proofs of some known (or essentially known) results.

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1. C^* -ALGEBRAS AND THEIR PRIMITIVE IDEAL SPECTRUM

We begin with a review of some needed general C^* -algebra results. We recall [14] that a C^* -algebra is a complex algebra A together with a conjugate linear

involution $*$ and a complete norm $\| \cdot \|$ such that $(ab)^* = b^*a^*$, $\|ab\| \leq \|a\|\|b\|$, and $\|a^*a\| = \|a\|^2$, for all $a, b \in A$. (The fact that $*$ is an involution means that $a^{**} = a$.) In particular, a C^* -algebra is also a Banach algebra. Let \mathcal{H} be a Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the space of linear, bounded operators on \mathcal{H} . One of the main reasons why C^* -algebras are important in applications is that every norm-closed subalgebra $A \subset \mathcal{L}(\mathcal{H})$ that is also closed under taking Hilbert space adjoints is a C^* -algebra. Abstract C^* -algebras have many non-trivial properties that can then be used to study the concretely given algebra A . Conversely, every abstract C^* -algebra is isometrically isomorphic to a norm closed subalgebra of $\mathcal{L}(\mathcal{H})$ (the Gelfand-Naimark theorem, see [14, theorem 2.6.1]). A *representation* of a C^* -algebra A on the Hilbert space \mathcal{H}_π is a morphism $\pi : A \rightarrow \mathcal{L}(\mathcal{H}_\pi)$ to the algebra of bounded operators on \mathcal{H}_π . (Recall that, in this paper, by a morphism of C^* -algebras, we shall always mean a *$*$ -morphism*.) We shall use the fact that every morphism ϕ of C^* -algebras (and hence any representation of a C^* -algebra) has norm $\|\phi\| \leq 1$. Consequently, every bijective morphism of C^* -algebras is an isometric isomorphism, and, in particular

$$(3) \quad \|\phi(a)\| = \|a + \ker(\phi)\|_{A/\ker(\phi)}.$$

A two-sided ideal $I \subset A$ is called *primitive* if it is the kernel of an irreducible representation. We shall denote by $\text{Prim}(A)$ the set of primitive ideals of A . For any two-sided ideal $J \subset A$, we have that its primitive ideal spectrum $\text{Prim}(J)$ identifies with the set of all the primitive ideals of A *not* containing the two-sided ideal $J \subset A$. It turns out then that the sets of the form $\text{Prim}(J)$, where J ranges through the set of two-sided ideals $J \subset A$, define a topology on $\text{Prim}(A)$, called the *Jacobson topology* on $\text{Prim}(A)$. If $A = C(K)$, the algebra of continuous functions on a compact space K , then K and $\text{Prim}(A)$ are canonically homeomorphic. See Example 4.5 for a slightly more involved example.

Throughout this paper, we shall denote by A a generic C^* -algebra. Also, by $\phi : A \rightarrow \mathcal{L}(\mathcal{H}_\phi)$ we shall denote generic representations of A . For any representation ϕ of A , we define its support, $\text{supp}(\phi) \subset \text{Prim}(A)$ as the complement of $\text{Prim}(\ker(\phi))$, that is, $\text{supp}(\phi) := \text{Prim}(A) \setminus \text{Prim}(\ker(\phi))$ is the set of primitive ideals of A *containing* $\ker(\phi)$.

Remark 1.1. The irreducible representations of A do not form a set (there are too many of them). The *unitary equivalence classes* of irreducible representations of A do form a set however, which we shall denote by \hat{A} . By $\pi : A \rightarrow \mathcal{L}(\mathcal{H}_\pi)$ we shall denote an arbitrary *irreducible* representation of A . There exists then by definition a surjective map

$$(4) \quad \text{can} : \hat{A} \rightarrow \text{Prim}(A)$$

that associates to (the class of) each irreducible representation $\pi \in \hat{A}$ its kernel $\ker(\pi)$. For each $a \in A$ and each irreducible representation π of A , the algebraic properties of $\pi(a)$ depend only on the kernel of π . That yields a well defined function

$$(5) \quad \text{can} : \hat{A} \ni \pi \rightarrow \|\pi(a)\| \in [0, \|a\|],$$

which descends to a well defined function

$$(6) \quad n_a : \text{Prim}(A) \ni \pi \rightarrow \|\pi(a)\| \in [0, \|a\|], \quad n_a(\ker(\pi)) = \|\pi(a)\|,$$

because if ϕ_1 and ϕ_2 are representations of A with the same kernel, then $\|\phi_1(a)\| = \|\phi_2(a)\|$ for all $a \in A$.

A C^* -algebra is *type I* if, and only if, the surjection $\text{can} : \hat{A} \rightarrow \text{Prim}(A)$ of Equation (4) is, in fact, a bijection [14] (a deep result). Then the discussion of Remark 1.1 becomes unnecessary and several arguments below will be (slightly) simplified since we will not have to make distinction between equivalence classes of irreducible representations and their kernels. Fortunately, many (if not all) of the C^* -algebras that arise in the study of pseudodifferential operators and of other practical questions are type I C^* -algebras. In spite of this, it seems unnatural at this time to restrict our study to type I C^* -algebras. Therefore, we will not assume that A is a type I C^* -algebra, unless this assumption is really needed. When A is a type I C^* -algebra, we will identify \hat{A} and $\text{Prim}(A)$.

We shall need the following simple (and well known) lemma [14].

Lemma 1.2. *The map $n_a : \text{Prim}(A) \ni I \rightarrow \|a + I\|_{A/I} \in [0, \|a\|]$ is lower semi-continuous, that is, the set $\{I \in \text{Prim}(A), \|a + I\|_{A/I} > t\}$ is open for any $t \in \mathbb{R}$.*

We include the simple proof for the benefit of the non-specialist.

Proof. Let us fix $t \in \mathbb{R}$. Since n_a takes on non-negative values, we may assume $t \geq 0$. Let then $\chi : [0, \infty) \rightarrow [0, 1]$ be a continuous function that is zero on $[0, t^2]$ but is > 0 on (t^2, ∞) and let $b = \chi(a^*a)$, which is defined using the functional calculus with continuous functions. If $\phi : A \rightarrow \mathcal{L}(\mathcal{H}_\phi)$ is a representation of A , then we have that $\|\phi(a)\|^2 = \|\phi(a^*a)\| \leq t^2$ if, and only if,

$$\chi(\phi(a^*a)) = \phi(\chi(a^*a)) = \phi(b) = 0.$$

Let then J be the (closed) two sided ideal generated by b , that is, $J := \overline{AbA}$. Then

$$\begin{aligned} \{I \in \text{Prim}(A), \|a + I\|_{A/I} \leq t\} &= \{I \in \text{Prim}(A), b \in I\} \\ &= \{I \in \text{Prim}(A), J \subset I\} = \text{Prim}(A) \setminus \text{Prim}(J), \end{aligned}$$

is hence a closed set. Consequently, $\{I \in \text{Prim} A, \|a + I\|_{A/I} > t\}$ is open, as claimed. \square

2. FAITHFUL FAMILIES

Let \mathcal{F} be a set of representations of A . We say that the family \mathcal{F} is *faithful* if the direct sum representation $\rho := \oplus_{\phi \in \mathcal{F}} \phi$ is injective. Faithful families of irreducible representations of a C^* -algebra A were called *weakly sufficient* in [41]. The results of this subsection are for the most part very well-known, see for instance [41], but we include them for the purpose of later reference and in order to compare them with the properties of exhausting families and strictly norming families. We have the following well known result that will serve us as a model for characterization of “strictly norming families” of representations in the next subsection.

Proposition 2.1. *Let \mathcal{F} be a family of representations of the C^* -algebra A . The following are equivalent:*

- (i) *The family \mathcal{F} is faithful.*
- (ii) *The union $\cup_{\phi \in \mathcal{F}} \text{supp}(\phi)$ is dense in $\text{Prim}(A)$.*
- (iii) *$\|a\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\|$ for all $a \in A$.*

Proof. (i) \Rightarrow (ii). We proceed by contradiction. Let us assume that (i) is true, but that (ii) is not true. That is, we assume that $\cup_{\phi \in \mathcal{F}} \text{supp}(\phi)$ is not dense in $\text{Prim}(A)$. Then there exists a non empty open set $\text{Prim}(J) \subset \text{Prim}(A)$ that does not intersect $\cup_{\phi \in \mathcal{F}} \text{supp}(\phi)$, where $J \subset A$ is a non-trivial two-sided ideal. Then $J \neq 0$ is contained in the kernel of $\oplus_{\phi \in \mathcal{F}} \phi$ and hence \mathcal{F} is not faithful. This is a contradiction, and hence (ii) must be true if (i) is true.

(ii) \Rightarrow (iii). For a given $a \in A$, the map sending the kernel $\ker \pi$ of an irreducible representation π to $\|\pi(a)\|$ is a lower semi-continuous function $\text{Prim}(A) \rightarrow [0, \infty)$, by Lemma 1.2. Moreover, for any $a \in A$ there exists an irreducible representation π_a such that $\|\pi_a(a)\| = \|a\|$. Hence, for every $\epsilon > 0$, $\{\pi \in \text{Prim}(A), \|\pi(a)\| > \|a\| - \epsilon\}$ is a non empty open set (it contains $\ker \pi_a$) and then it contains some $\pi \in \cup_{\phi \in \mathcal{F}} \text{supp}(\phi)$, since the later set was assumed to be dense in $\text{Prim}(A)$. Let $\phi \in \mathcal{F}$ be such that $\ker(\pi) \supset \ker(\phi)$. Then

$$\|a\| \geq \|\phi(a)\| \geq \|\pi(a)\| > \|a\| - \epsilon,$$

where the first inequality is due to the general fact that representations of C^* -algebras have norm ≤ 1 and the second one is due to the fact that

$$\|\phi(a)\| = \|a + \ker(\phi)\|_{A/\ker(\phi)} \geq \|a + \ker(\pi)\|_{A/\ker(\pi)} = \|\pi(a)\|,$$

by Equation (3). Consequently, $\|a\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\|$, as desired.

(iii) \Rightarrow (i). Let $\rho := \oplus_{\phi \in \mathcal{F}} \phi : A \rightarrow \oplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$. We need to show that ρ is injective. The norm on $\oplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$ is the sup norm, that is, $\|(T_\phi)_{\phi \in \mathcal{F}}\| = \sup_{\phi \in \mathcal{F}} \|T_\phi\|$. Therefore $\|\rho(a)\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\| = \|a\|$, since we are assuming (iii). Consequently, ρ is isometric, and hence it is injective. \square

In the next proposition we shall need to assume that A is unital (that is, that it has a unit $1 \in A$). This assumption is not very restrictive since, given any non-unital C^* -algebra A_0 , the algebra with adjoint unit $A = A_0^+ := A_0 \oplus \mathbb{C}$ has a unique C^* -algebra norm. For any unital C^* -algebra A and any $a \in A$, we denote by $\text{Spec}_A(a)$ the *spectrum* of a in A , defined by

$$\text{Spec}_A(a) := \{ \lambda \in \mathbb{C}, \lambda - a \text{ is not invertible in } A \}.$$

is known that $\text{Spec}_A(a)$ is, in fact, independent of the C^* -algebra A [14]. (See next.) It is also known classically that $\text{Spec}_A(a)$ is compact and non-empty, unlike in the case of unbounded operators [14]. For A non-unital, we let $\text{Spec}(a) := \text{Spec}_{A^+}(a)$.

We shall need the following general property of C^* -algebras [14].

Lemma 2.2. *Let $A_1 \subset B$ be two C^* -algebras and $a \in A_1$ be such that it has an inverse in B , denoted a^{-1} . Then $a^{-1} \in A_1$. In particular, the spectrum of a is independent of the C^* -algebra in which we compute it:*

$$(7) \quad \text{Spec}_{A_1}(a) = \text{Spec}_B(a) =: \text{Spec}(a).$$

We shall need the following remark on extensions of representations.

Remark 2.3. Let B be a C^* -algebra and $I \subset B$ be a closed two-sided ideal. Recall from Proposition 2.10.4 in [14] that any representation $\pi : I \rightarrow \mathcal{L}(\mathcal{H})$ extends to a unique representation $\pi : B \rightarrow \mathcal{L}(\mathcal{K}) \subset \mathcal{L}(\mathcal{H})$, $\mathcal{K} = \overline{\pi(I)\mathcal{H}}$ (the closure is actually not needed by the Cohen-Hewitt factorization theorem). This extension is an instance of the Rieffel induction [40] corresponding to I , regarded as an A - I bimodule.

In particular, we shall use this remark in order to deal with non-unital algebras as follows.

Notations 2.4. Let I be a C^* -algebra and let us denote by $I' := I$ if I has a unit and by $I' := I^+ := I \oplus \mathbb{C}$ if I does not have a unit. Let $\chi_0 : I^+ \rightarrow \mathbb{C}$ be the canonical projection. Then, if \mathcal{F} is a set of representations of I , we let $\mathcal{F}' := \mathcal{F}$ if I has a unit and $\mathcal{F}' := \mathcal{F} \cup \{\chi_0\}$ if I does not have a unit. By implicitly extending the representations of I to I' , we have that \mathcal{F}' is a set of representations of I^+ .

Using this notation, we have the following result.

Proposition 2.5. *Let \mathcal{F} be a faithful family of nondegenerate representations of a C^* -algebra A . An element $a \in A'$ is invertible if, and only if, $\phi(a)$ is invertible in $\mathcal{L}(\mathcal{H}_\phi)$ for all $\phi \in \mathcal{F}'$ and the set $\{\|\phi(a)^{-1}\|, \phi \in \mathcal{F}'\}$ is bounded.*

Proof. By replacing A with A' , we may assume that A is unital. Since each $\phi \in \mathcal{F}$ is nondegenerate, if a is invertible, $\phi(a)$ also is invertible and $\|\phi(a)^{-1}\| = \|\phi(a^{-1})\| \leq \|a^{-1}\|$ is hence bounded.

Conversely, let ρ be the direct sum of all the representations $\phi \in \mathcal{F}$, that is,

$$(8) \quad \rho := \bigoplus_{\phi \in \mathcal{F}} \phi : A \longrightarrow \bigoplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi).$$

If $\|\phi(a)\|$ is invertible for all $\phi \in \mathcal{F}$ and there exists M independent of ϕ such that $\|\phi(a)^{-1}\| \leq M$, then $b := (\phi(a)^{-1})_{\phi \in \mathcal{F}}$ is a well defined element in $B := \bigoplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$ and b is an inverse for $\rho(a)$ in B . Let $A_1 := \rho(A)$. Then $\rho(a) \in A_1$ is invertible in B . Then observe that since ρ is continuous, injective, and surjective morphism of C^* -algebras, it defines an isomorphism of algebras $A \rightarrow A_1$. We then conclude that a is invertible in A as well. \square

The following is a converse of the above proposition. Recall that $a \in A$ is called *normal* if $aa^* = a^*a$.

Proposition 2.6. *Let \mathcal{F} be a family of representations of a unital C^* -algebra A with the following property:*

“If $a \in A$ is such that $\phi(a)$ is invertible in $\mathcal{L}(H_\phi)$ for all $\phi \in \mathcal{F}$ and the set $\{\|\phi(a)^{-1}\|, \phi \in \mathcal{F}\}$ is bounded, then a is invertible in A .”

Then the family \mathcal{F} is faithful.

Proof. Clearly, the family \mathcal{F} is not empty, since otherwise all elements of A would be invertible, which is not possible. Let us assume, by contradiction, that the family \mathcal{F} is *not* faithful. Then, by Proposition 2.1(ii), there exists a non-empty open set $V \subset \text{Prim}(A)$ that does not intersect $\bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$. Let $J \subset A$, $J \neq 0$, be the (closed) two-sided ideal corresponding to V , that is, $V = \text{Prim}(J)$. Since \mathcal{F} is non-empty, we have $J \neq \text{Prim}(A)$. Then every $\phi \in \mathcal{F}$ is such that $\phi = 0$ on J . Let $a \in J$, $a \neq 0$. By replacing a with $a^*a \in J$, we can assume $a \geq 0$. Let $\lambda \in \text{Spec}(a)$, $\lambda \neq 0$. Such a λ exists since a is normal and non-zero. Let $c := \lambda - a$. Then, for any $\phi \in \mathcal{F}$, $\phi(c) = \lambda \in \mathbb{C}$ is invertible and $\|\phi(c)^{-1}\| = \lambda^{-1}$ is bounded. However, c is not invertible (in any C^* -algebra containing it) since it belongs to the non-trivial ideal J . \square

Recall that $\mathcal{C}_0(X)$ is the set of continuous functions on X that have vanishing limit at infinity. Then $\mathcal{C}_0(X)$ is a commutative C^* -algebra, and all commutative C^* -algebras are of this form.

Example 2.7. Let μ_α , $\alpha \in I$, be a family of positive, regular Borel measures on a locally compact space X . Let ϕ_α be the corresponding multiplication representation of the C^* -algebra $\mathcal{C}_0(X) \rightarrow \mathcal{L}(L^2(X, \mu_\alpha))$. We have $\text{supp}(\phi_\alpha) = \text{supp}(\mu_\alpha)$ and the family $\mathcal{F} := \{\phi_\alpha, \alpha \in I\}$ is faithful if, and only if, $\bigcup_{\alpha \in I} \text{supp}(\mu_\alpha)$ is dense in X . In particular, if each μ_α is the Dirac measure concentrated at some $x_\alpha \in X$, then $\phi_\alpha(f) = f(x_\alpha) =: \text{ev}_{x_\alpha}(f) \in \mathbb{C}$ and $\text{supp}(\mu_\alpha) = \{x_\alpha\}$. We shall henceforth identify $x_\alpha \in X$ with the corresponding evaluation irreducible representation ev_{x_α} . Then we have that

$$\mathcal{F} = \{\text{ev}_{x_\alpha}, \alpha \in I\} \text{ is faithful} \Leftrightarrow \{x_\alpha, \alpha \in I\} \text{ is dense in } X.$$

This example extends right away to C^* -algebras of the form $\mathcal{C}_0(X; \mathcal{K})$ of functions with values compact operators on some given Hilbert space.

We conclude our discussion of faithful families with the following result. We denote by $\overline{\bigcup_{\alpha} S_\alpha} := \overline{\bigcup_{\alpha} S_\alpha}$ the closure of the union of the family of sets S_α .

Proposition 2.8. *Let \mathcal{F} be a family of representations of a unital C^* -algebra A . Then \mathcal{F} is faithful if, and only if, for any normal $a \in A$,*

$$(9) \quad \text{Spec}(a) = \overline{\bigcup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}.$$

Proof. Let us assume first that the family \mathcal{F} is faithful and that a is normal. Since we have that $\text{Spec}(\phi_0(a)) \subset \text{Spec}(a)$ for any representation ϕ_0 of A , it is enough to show that $\text{Spec}(a) \subset \overline{\bigcup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}$. Let us assume the contrary and let $\lambda \in \text{Spec}(a) \setminus \overline{\bigcup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}$. By replacing a with $a - \lambda$, we can assume that $\lambda = 0$. We thus have that $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$, but a is not invertible (in A). Moreover, $\|\phi(a)^{-1}\| \leq \delta^{-1}$, where δ is the distance from $\lambda = 0$ to the spectrum of $\phi(a)$, by the properties of the functional calculus for normal operators. This is however a contradiction by Proposition 2.5, which implies that a must be invertible in A as well.

To prove the converse, let us assume that $\text{Spec}(a) \subset \overline{\bigcup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}$, for all normal elements $a \in A$. Let J be a non-trivial (closed *selfadjoint*) two-sided ideal on which all the representations $\phi \in \mathcal{F}$ vanish. We have to show that $J = 0$, which would imply that \mathcal{F} is faithful. Let $a \in J$ be a normal element. Then $\text{Spec}(a) \subset \overline{\bigcup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))} = \{0\}$. Since a is normal we deduce $a = 0$ and hence J has no normal element other than 0. Then, for any $a \in J$, we can write $a = 1/2(a + a^*) + 1/2(a - a^*)$, the sum of two normal elements in J because J is selfadjoint. Therefore $1/2(a + a^*) = 1/2(a - a^*) = 0$, and hence $a = 0$ and $J = 0$. \square

We refer to [4, 6, 25, 32, 38, 46] for background material on groupoids. The following is well known, but is useful in order to set up the terminology and to introduce some concepts to be used below.

Example 2.9. Let \mathcal{G} be a locally compact groupoid with units M and with Haar system (λ_x) , $x \in M$. If $d : \mathcal{G} \rightarrow M$ denotes the domain map $\mathcal{G} \rightarrow M$, then we denote $\mathcal{G}_A := d^{-1}(A)$, $A \subset M$, and $\mathcal{G}_x := d^{-1}(x)$, $x \in M$. We recall that λ_x has support \mathcal{G}_x (and is right invariant and continuous in a natural sense). The regular representation π_x of $C^*(\mathcal{G})$ then acts on $L^2(\mathcal{G}_x, \lambda_x)$ by left convolution. Let $\mathcal{R} := \{\pi_x, x \in M\}$ be the set of regular representations of $C^*(\mathcal{G})$, the C^* -algebra associated to \mathcal{G} . Let I be the intersection of all the kernels of the representations π_x . Then the set \mathcal{R} is a faithful set of representations of $C_r^*(\mathcal{G}) \simeq C^*(\mathcal{G})/I$, the reduced

C^* -algebra of \mathcal{G} . In general, \mathcal{R} will not be a faithful family of representations of $C_r^*(\mathcal{G})$, unless the canonical projection $C^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G})$ is an isomorphism.

3. EXHAUSTING AND STRICTLY NORMING FAMILIES

Let us notice that Example 2.7 shows that the ‘sup’ in the relation $\|a\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\|$ (Proposition 2.1) may not be attained. It also shows that the closure of the union in Equation (9) is needed. Sometimes, in applications, one does obtain however the stronger version of these results (that is, that the sup is attained and that the closure is not needed), see [9, 19], for example. Moreover, the condition that the norms of $\phi(a)^{-1}$ be uniformly bounded (in ϕ) for any fixed $a \in A$ is inconvenient and often not needed in applications. For this reason, we introduce now a new class of sets of representations of A , the class of “exhausting sets of representations,” a class that has some additional properties. The concept of an exhausting set of representations turns out to be closely related to the concept of an “invertibility sufficient set of representations”, introduced by Roch [41], which we discuss first.

3.1. Invertibility sufficient sets of representations. We now recall the concepts of invertibility sufficient and strictly norming families of representations [17, 41]. See also [37, 42].

Definition 3.1 (Roch). Let \mathcal{F} be a set of representations of a *unital* C^* -algebra A .

(i) We shall say that \mathcal{F} is *invertibility sufficient* if

$$“a \in A \text{ is invertible} \Leftrightarrow \phi(a) \text{ is invertible for any } \phi \in \mathcal{F}.”$$

(ii) We shall say that \mathcal{F} is *strictly norming* if, for any $a \in A$, there exists $\phi \in \mathcal{F}$ such that $\|a\| = \|\phi(a)\|$.

Example 3.2. By classical results [14], the set of all irreducible representations of a C^* -algebra is strictly norming. A proof of this well-known fact is contained in [17]. See also Theorem 3.4.

The classes of invertibility sufficient and strictly norming sets of representations actually coincide (see Theorem 3.4 below). Before discussing that result, however, we need to extend the above definitions to the non-unital case.

Remark 3.3. Using the notation introduced in 2.4, we obtain then the following form of the definition of an invertibility sufficient family:

“The family \mathcal{F} is *invertibility sufficient* if $1 + a \in A^+ := A \oplus \mathbb{C}$, $a \in A$, is invertible if, and only if, $1 + \phi(a)$ is invertible for any $\phi \in \mathcal{F}$.”

Similarly, the definition of a strictly norming family becomes:

“ \mathcal{F} is *strictly norming* if, for any $a \in A$ and $\lambda \in \mathbb{C}$, either there exists $\phi \in \mathcal{F}$ such that $\|\lambda + a\| = \|\lambda + \phi(a)\|$ or $\|\lambda + a\| = |\lambda|$.”

The following result was proved in the unital case in [41]. See also [17].

Theorem 3.4 (Roch). *Let \mathcal{F} be a set of non-degenerate representations of a unital C^* -algebra A . Then \mathcal{F} is strictly norming if, and only if, it is invertibility sufficient.*

Proof. The unital case was proved already. If A does not have a unit, then we simply replace A with A' and \mathcal{F} with \mathcal{F}' (see the notation introduced in 2.4) to reduce to the unital case. \square

Clearly, an invertibility sufficient family of representations will consist only of non-degenerate representations, but this is not true of a strictly norming family.

We now give some examples of how exhausting and strictly norming sets of representations are useful for invertibility questions. The following characterization of Fredholm operators is a consequence of the definitions.

Corollary 3.5. *Let $1 \in A \subset \mathcal{L}(\mathcal{H})$ be a sub- C^* -algebra of bounded operators on the Hilbert space \mathcal{H} containing the algebra of compact operators on \mathcal{H} , $\mathcal{K} = \mathcal{K}(\mathcal{H})$. Let \mathcal{F} be an invertibility sufficient family of representations of A/\mathcal{K} . We then have the following characterization of Fredholm operators $a \in A$:*

$a \in A$ is Fredholm if, and only if, $\phi(a)$ is invertible in for all $\phi \in \mathcal{F}$.

The following proposition is the analog of Proposition 2.8 in the framework of strictly norming families.

Theorem 3.6. *Let \mathcal{F} be a family of representations of a unital C^* -algebra A . Then \mathcal{F} is invertibility sufficient if, and only if, for any $a \in A$,*

$$(10) \quad \text{Spec}(a) = \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a)).$$

Proof. Let us assume first that the family \mathcal{F} is invertibility sufficient. We proceed in analogy with the proof of Proposition 2.8. Since we have that $\text{Spec}(\phi_0(a)) \subset \text{Spec}(a)$ for any representation ϕ_0 of A , it is enough to show that $\text{Spec}(a) \subset \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$. Let us assume the contrary and let $\lambda \in \text{Spec}(a) \setminus \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$. By replacing a with $a - \lambda$, we can assume that $\lambda = 0$. We thus have that $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$, but a is not invertible (in A), contradicting the assumption that \mathcal{F} is invertibility sufficient.

To prove the converse, let us assume that $\text{Spec}(a) \subset \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$ for all $a \in A$. Let us assume that $a \in A$ and that $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$. Then $0 \notin \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$. Since $\text{Spec}(a) \subset \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$, we have that $0 \notin \text{Spec}(a)$, and hence a is invertible. Thus the family \mathcal{F} is invertibility sufficient. \square

3.2. Exhausting families of representations. It is not always easy to check that a family of representations is invertibility sufficient (or strictly norming, for that matter). For this reason, we introduce a slightly more restrictive class of families of representations, the class of exhausting families of representations. It is convenient to do this for ideals first.

Definition 3.7. Let A be a C^* algebra, possibly without unit, and let \mathcal{I} a set of (closed, two-sided) ideals $I \subset A$. We say that \mathcal{I} is *exhausting* if, by definition, for any irreducible representation π of A , there exists $I \in \mathcal{I}$ such that $I \subset \ker(\pi)$.

We shall typically work with families of representations \mathcal{F} . We consider, nevertheless, the case of families of morphisms as well. We thus have the following closely related definition.

Definition 3.8. Let \mathcal{F} be a set of morphisms $\phi : A \rightarrow B_\phi$ of a (not necessarily unital) C^* -algebra A . The algebras B_ϕ are not fixed. We shall say that \mathcal{F} is *exhausting* if the family of ideals $\{\ker(\phi), \phi \in \mathcal{F}\}$ is exhausting. Similarly, a set of unitary

equivalence classes of representations \mathcal{F} of A is exhausting if the corresponding set of kernels is exhausting.

The following simple remark is sometimes useful.

Remark 3.9. Let ϕ be a representation of A . Recall that $\text{supp}(\phi)$ is the set of primitive ideals of A that contain $\ker(\phi)$. Moreover, $\ker(\phi)$ depends only on the unitary equivalence class of ϕ . We then see that \mathcal{F} is exhausting if, and only if, $\text{Prim}(A) = \cup_{\phi \in \mathcal{F}} \text{supp}(\phi)$.

Recall that we denote by $A' := A$ if A has a unit and $A' := A^+ := A \oplus \mathbb{C}$, the algebra of adjoint unit, if A does not have a unit.

Proposition 3.10. *Let A be a possibly non-unital C^* -algebra and let \mathcal{F} be a family of representations of A . We denote by $\mathcal{F}' = \mathcal{F}$ if A has a unit and by $\mathcal{F}' := \mathcal{F} \cup \{\chi_0\}$, where $\chi_0 : A' = A \oplus \mathbb{C} \rightarrow \mathbb{C}$ is the canonical projection (as in 2.4). Then we have*

- (i) *\mathcal{F} is an exhausting set of representations of A if, and only if, \mathcal{F}' is an exhausting set of representations of A' .*
- (ii) *\mathcal{F} is an invertibility sufficient set of representations of A if, and only if, \mathcal{F}' is an invertibility sufficient set of representations of A' .*

Proof. To prove (i), we only need to consider the case when A does not have a unit. The result then follows from Remark 3.9 and from the relation $\text{Prim}(A') = \text{Prim}(A^+) = \text{Prim}(A) \cup \{\ker(\chi_0)\}$, where, we recall, $\ker(\chi_0) = A$. The other statement is really the corresponding definitions. \square

Remark 3.11. Let \mathcal{F}_i , $i = 1, 2$, be two families of representations of A . Let denote by $\mathcal{I}_i := \{\ker(\phi), \phi \in \mathcal{F}_i\}$. We assume that $\mathcal{I}_1 = \mathcal{I}_2$. Then the families \mathcal{F}_i are at the same time exhausting or not. The same is true for the properties of being strictly norming, or invertibility sufficient. So these properties are really properties of a family of ideals of A rather than of families of representations of A . Nevertheless, it is customary to work with families of representation rather than families of ideals. In the same way, we can consider the analogous properties of families of *morphisms* of C^* -algebras.

Let us record the following simple facts, for further use.

Proposition 3.12. *Let \mathcal{F} be a set of representations of a C^* -algebra. If \mathcal{F} is exhausting, then \mathcal{F} is invertibility preserving and hence also strictly norming. If \mathcal{F} is strictly norming, then it is also faithful.*

Proof. Let A be the given C^* -algebra. Let us prove first that any exhausting family \mathcal{F} is strictly norming. Indeed, let $a \in A'$. Then there exists an irreducible representation π of A' such that $\|\pi(a)\| = \|a\|$ [14]. Unless $1 \notin A$ and $\pi = \chi_0$, where $\chi_0 : A' = A \oplus \mathbb{C} \rightarrow \mathbb{C}$ is the projection, there will exist $\phi \in \mathcal{F}$ such that $\pi \in \text{supp}(\phi)$. Then, as in the proof of (ii) \Rightarrow (iii) in Proposition 2.1, we have that $\|a\| = \|\pi(a)\| \leq \|\phi(a)\| \leq \|a\|$. Hence $\|\phi(a)\| = \|a\|$. On the other hand, if $1 \notin A$ and $\pi = \chi_0$, then let $a = \lambda + a_0$, with $\lambda \in \mathbb{C}$ and $a_0 \in A$. Then $\|\lambda + a_0\| = \|a\| = \|\pi(a)\| = |\lambda|$. Since any strictly norming family is invertibility preserving, by Theorem 3.4, the first part of the proposition follows.

Let us prove first that any strictly norming family \mathcal{F} is faithful. Indeed, let us consider the representation $\rho := \oplus_{\phi \in \mathcal{F}} \phi : A \rightarrow \oplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$. By the definition of a strictly norming family of representations, the representation ρ is isometric. Therefore it is injective and consequently \mathcal{F} is faithful. \square

We summarize the above Proposition in

$$\mathcal{F} \text{ exhausting} \Rightarrow \mathcal{F} \text{ strictly norming} \Rightarrow \mathcal{F} \text{ faithful.}$$

In the next two examples we will see that there exist faithful families that are not strictly norming and strictly norming families that are not exhausting.

Example 3.13. We consider again the framework of Example 2.7 and consider only families of *irreducible representations*. Thus $A = \mathcal{C}_0(X)$, for a locally compact space X . The irreducible representations of A then identify with the points of X , since $X \simeq \text{Prim}(A) = \hat{A}$. A family \mathcal{F} of *irreducible* representations of A thus identifies with a subset $\mathcal{F} \subset X$. We then have that a family $\mathcal{F} \subset X$ of irreducible representations of $A = \mathcal{C}_0(X)$ is faithful if, and only if, \mathcal{F} is *dense* in X . On the other hand, a family of irreducible representations of $A = \mathcal{C}_0(X)$ is exhausting if, and only if, $\mathcal{F} = X$.

The relation between exhausting and strictly norming families is not so simple. We begin with the following remark on the above example.

Remark 3.14. If in Example 3.13 X is moreover *metrisable*, then every strictly norming family $\mathcal{F} \subset X$ is also exhausting, because for any $x \in X$, there exists a compactly supported, continuous function $\psi_x : X \rightarrow [0, 1]$ such that $\psi_x(x) = 1$ and $\psi_x(y) < 1$ for $y \neq x$ (we can do that by arranging that $\psi_x(y) = 1 - d(x, y)$, for $d(x, y)$ small, and use the Tietze extension theorem. In general, however, it is not true that any strictly norming family is exhausting. Indeed, let I be an uncountable set and $X = [0, 1]^I$. Let $x \in X$ be arbitrary, then the family $\mathcal{F} := X \setminus \{x\}$ is strictly norming but is not exhausting. Indeed, let $f : X \rightarrow [0, 1]$ be a continuous function such that $f(x) = 1$. Since f depends on a countable number of variables, the set $\{f = 1\}$ will not be reduced to x alone. See also Theorem 4.4.

We conclude this subsection with the following result that is relevant for the next subsection. See also [42] and the comment at the end of this subsection. The results in that book can be used to give a quick proof of the following results for invertibility sufficient families (which are essentially contained in that book). For the benefit of the reader, we include nevertheless the short, direct proofs, since we are also interested in exhausting families.

Proposition 3.15. *Let $I \subset A$ be an ideal of a C^* -algebra. Let \mathcal{F}_I be a set of nondegenerate representations of I and $\mathcal{F}_{A/I}$ be a set of representations of A/I . Let $\mathcal{F} := \mathcal{F}_I \cup \mathcal{F}_{A/I}$, regarded as a family of representations of A . If \mathcal{F}_I and $\mathcal{F}_{A/I}$ are both exhausting, then \mathcal{F} is also exhausting. The same result holds by replacing exhausting with strictly norming.*

Proof. We have that $\text{Prim}(A)$ is the disjoint union of $\text{Prim}(I)$ and $\text{Prim}(A/I)$. Since $\cup_{\phi \in \mathcal{F}_I} \text{supp}(\phi) \subset \text{Prim}(I)$ and $\cup_{\phi \in \mathcal{F}_{A/I}} \text{supp}(\phi) \subset \text{Prim}(A/I)$, the result about exhausting families follows from the definition.

Let us assume that both \mathcal{F}_I and $\mathcal{F}_{A/I}$ are strictly norming and let $a \in A$. We may assume that A is unital. We want to show that \mathcal{F} is also strictly norming, that is, that there exists $\phi \in \mathcal{F}_I \cup \mathcal{F}_{A/I}$ such that $\|a\| = \|\phi(a)\|$. By replacing a with a^*a , we can assume that $a \geq 0$. Since $\mathcal{F}_{A/I}$ is strictly norming, there is $\phi \in \mathcal{F}_{A/I}$ such that $\|a + I\|_{A/I} = \|\phi(a)\|$. If $\|a + I\|_{A/I} = \|a\|$, we are done. Otherwise, let ψ be a continuous function on $\text{Spec}(a)$ that is zero on $\text{Spec}_{A/I}(a + I)$ and such that

$\psi(\|a\|) = \|a\|$ and $\psi(t) \leq t$ for $t \geq 0$. Then $\psi(a) \in I$ and $\|\psi(a)\| = \|a\|$. Since the family \mathcal{F}_I is strictly norming, there exists $\phi \in \mathcal{F}_I$ such that

$$\|a\| = \|\psi(a)\| = \|\phi(\psi(a))\| = \|\psi(\phi(a))\| \leq \|\phi(a)\|$$

This shows that the family \mathcal{F} is strictly norming. \square

We have the following consequence that is sometimes useful in applications.

Corollary 3.16. *Let $I \subset A$ be a two-sided ideal in a C^* -algebra A . Let \mathcal{F} be an invertibility preserving family of representations of I . Then $a \in A$ is invertible if, and only if, a is invertible in A/I and $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$.*

Proof. Since \mathcal{F} is an invertibility preserving set of representations of I , it consists of non-degenerate representations, which will hence extend uniquely to A . Let π be an isometric representation of A/I . The result then follows from Proposition 3.15 applied to $\mathcal{F}_I := \mathcal{F}$ and $\mathcal{F}_{A/I} := \{\pi\}$. \square

Results closely related to Proposition 3.15 and Corollary 3.16 were obtained in [42] under the name of “lifting theorems.” See especially Section 6.3 of that book. The results in that book were typically obtained in a more general setting: often using ideals in a Banach algebra and sometimes using even general ideals (and morphisms). The interested reader should consult that book as well.

3.3. Groupoid algebras and the Effros-Hahn conjecture. We now show how one can check in the framework of locally compact groupoids (with additional properties) that certain families of representations are exhausting, thus generalizing some results of [17].

We refer to the Example 2.9 and, especially, to the references quoted before that example, for notations and results pertaining to groupoids. In particular, we shall denote by d and r the domain and range maps of a groupoid \mathcal{G} and by $\mathcal{G}_x^x := d^{-1}(x) \cap r^{-1}(x)$ the *isotropy group* of x . This is the group of arrows (or morphisms) of \mathcal{G} that have domain and range equal to the unit x . Also, we continue to denote by $\mathcal{R} := \{\pi_y, y \in M\}$ the set of regular representations of a locally compact groupoid \mathcal{G} with Haar system and units M . Recall that we denote $\mathcal{G}_A := d^{-1}(A)$, $A \subset M$, and $\mathcal{G}_x := d^{-1}(x)$, $x \in M$.

We shall say that a locally compact groupoid \mathcal{G} with a Haar system *has the generalized Effros-Hahn property* if every primitive ideal of $C^*(\mathcal{G})$ is induced from an isotropy subgroup \mathcal{G}_y^y of \mathcal{G} [20, 39]. (This should not be confused with the various “EH induction properties” introduced in [15].) We shall write $\text{Ind}_y^{\mathcal{G}}(\sigma)$ for the induced representation of $C^*(\mathcal{G})$ from the representation σ of \mathcal{G}_y^y . If \mathcal{G} has the generalized Effros-Hahn property and all the isotropy groups \mathcal{G}_y^y , $y \in M$ are amenable, we say that \mathcal{G} is *EH-amenable*.

Theorem 3.17. *Let \mathcal{G} be a locally compact groupoid with a Haar system and units M . If \mathcal{G} is EH-amenable, then the family $\mathcal{R} := \{\pi_y, y \in M\}$ of regular representations of $C^*(\mathcal{G})$ is exhausting. In particular, the family \mathcal{R} is invertibility sufficient and the canonical map $C^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G})$ is an isomorphism.*

Proof. Let I be any primitive ideal of $C^*(\mathcal{G})$. Then I is induced from the isotropy group \mathcal{G}_y^y , $y \in M$, by the assumption that \mathcal{G} has the generalized Effros-Hahn property. Since \mathcal{G}_y^y is amenable, every irreducible representation of \mathcal{G}_y^y is weakly contained in the regular representation ρ_y of \mathcal{G}_y^y . But $\text{Ind}_y^{\mathcal{G}}(\rho_y)$ is the regular representation π_y of $C^*(\mathcal{G})$ on $L^2(\mathcal{G}_y)$. Since induction preserves the weak containment of representations (see Proposition 6.26 of [40]), we obtain that I contains $\ker(\pi_y)$. This proves that the family $\mathcal{R} := \{\pi_y, y \in M\}$ is exhausting. Therefore \mathcal{R} is also faithful, and hence $C^*(\mathcal{G}) \simeq C_r^*(\mathcal{G})$ (see Example 2.9). The family \mathcal{R} is invertibility sufficient since it is exhausting (see Proposition 3.12). \square

We then obtain the following consequence.

Theorem 3.18. *Let \mathcal{G} be a locally compact groupoid with a Haar system and units M . If \mathcal{G} is Hausdorff, second countable, and (topologically) amenable, then the family $\mathcal{R} := \{\pi_y, y \in M\}$ is exhausting.*

Proof. Since \mathcal{G} is an amenable, Hausdorff, second countable, locally compact groupoid with a Haar system, we have that \mathcal{G} satisfies the Effros-Hahn conjecture by the main result in [20], that is, it has the generalized Effros-Hahn property. Since \mathcal{G} is amenable, all its isotropy groups \mathcal{G}_x^x are amenable [2]. The result then follows from Theorem 3.17. \square

This result extends a result of [17], who considered the case of étale groupoids. Let \mathcal{G} be a locally compact groupoid with a Haar system and units M . We notice, however, that the family $\mathcal{R} := \{\pi_y, y \in M\}$ of regular representations of the reduced C^* -algebra $C_r^*(\mathcal{G})$ of \mathcal{G} is not exhaustive in general, as can be seen from the following example.

Remark 3.19. Let G be the free group on two generators and let $K_n \subset G$, $n \in \mathbb{N}$, be decreasing sequence of normal subgroups of G of finite index with $\bigcap_{n=1}^{\infty} K_n = \{1\}$. Let us consider the family of groups $\mathcal{G} := \bigcup_n \{n\} \times G/K_n$, with $n \in \mathbb{N} \cup \{\infty\}$ and $K_{\infty} := \{1\}$. It is a groupoid with units $\mathbb{N} \cup \{\infty\}$. Its domain and range map are equal and equal to the projection onto the first component. The topology on $\mathcal{G}_{\mathbb{N}} := d^{-1}(\mathbb{N})$, the restriction of \mathcal{G} to \mathbb{N} , is discrete. A basis of the system of neighborhoods of (∞, g) is given by the sets $\{(n, gK_n), n \geq N\}$, where $N \geq 1$ is arbitrary ($g \in G$). We have that the trivial representation of G defines a representation χ of $C^*(\mathcal{G})$ supported at $\{\infty\}$. The trivial representation of G is the limit of the trivial representations of G/K_n , so it descends to a representation of $C_r^*(\mathcal{G})$. However, the trivial representation of G is not contained in the support of any of the representations λ_n , $n \in \mathbb{N} \cup \{\infty\}$, since G is not amenable. Thus the family of regular representations λ_n , $n \in \mathbb{N} \cup \{\infty\}$ is not exhaustive. This example is due to Voiculescu and it answers (in the negative) a question of Exel [17].

We are ready to prove now that the class of EH-amenable groupoids is closed under extensions and that suitable ideals and quotients of EH-amenable groupoids are also EH-amenable.

Proposition 3.20. *Let \mathcal{G} be a locally compact groupoid with a Haar system and units M . Let $U \subset M$ be an open \mathcal{G} -invariant subset and $F := M \setminus U$. We have that \mathcal{G} is EH-amenable if, and only if, \mathcal{G}_F and \mathcal{G}_U are EH-amenable.*

Proof. It is clear that the isotropy groups \mathcal{G}_x^x of \mathcal{G} are given by the isotropy groups of the restrictions \mathcal{G}_F and \mathcal{G}_U . This gives that all the isotropy groups of \mathcal{G} are

amenable if, and only if, the same property is shared by all the isotropy groups of the restrictions \mathcal{G}_F and \mathcal{G}_U .

Let us turn now to proving the induction property for the primitive ideals. We need the following general fact. Let A be a C^* -algebra and $J \subset A$ be a two-sided ideal, then we have that $\text{Prim}(A)$ is the disjoint union of $\text{Prim}(J)$ and $\text{Prim}(A/J)$ [14]. This correspondence sends a primitive ideal I of A to $I \cap J$, if $I \cap J \neq J$, and otherwise (i.e. if $J \subset I$) it sends I to I/J , which is an ideal of A/J .

We shall use this correspondence as follows. Let I be primitive ideal of $C^*(\mathcal{G})$. Since $C^*(\mathcal{G}_U)$ is an ideal of $C^*(\mathcal{G})$ and $C^*(\mathcal{G})/C^*(\mathcal{G}_U) \simeq C^*(\mathcal{G}_F)$, by a result of Renault [38, 39], we have that I corresponds uniquely to either a primitive ideal of $C^*(\mathcal{G}_F)$ or to a primitive ideal of $C^*(\mathcal{G}_U)$. We shall consider these two cases separately. Anticipating, the first case will correspond to induced representations from isotropy groups \mathcal{G}_y^y with $y \in F := M \setminus U$ and the second case will correspond to induced representations from isotropy groups \mathcal{G}_y^y with $y \in U$. We first notice that the restriction of the induced representation $\text{Ind}_y^{\mathcal{G}}(\sigma)$ of $C^*(\mathcal{G})$ (induced from the representation σ of \mathcal{G}_y^y) restricts to a non-zero representation of $C^*(\mathcal{G}_U)$ if, and only if, $y \in U$.

Let us then consider a primitive ideal $I \supset C^*(\mathcal{G}_U)$ of $C^*(\mathcal{G})$ and $I/C^*(\mathcal{G}_U)$ the corresponding ideal of $C^*(\mathcal{G}_F) \simeq C^*(\mathcal{G})/C^*(\mathcal{G}_U)$. Then I is induced from the irreducible representation σ of \mathcal{G}_y^y if, and only if, $y \in F$ and $I/C^*(\mathcal{G}_U)$ is induced from the irreducible representation σ of \mathcal{G}_y^y . This follows directly from the definition of induced representations [40]; in fact, the inducing module is the same for both ideals.

On the other hand, if the primitive ideal I of $C^*(\mathcal{G})$ does not contain $C^*(\mathcal{G}_U)$, then again we notice that I is induced from the irreducible representation σ of \mathcal{G}_y^y if, and only if, $y \in U$ and $I \cap C^*(\mathcal{G}_U)$ is induced from the irreducible representation σ of \mathcal{G}_y^y . This again follows from the results in [40], more precisely, from Induction in Stages Theorem 5.9 of that paper. Indeed, extending non-degenerate representations of an ideal to the whole algebra is a particular case of induction in stages (see the Remark 2.3). The inductions modules are again the same. \square

4. TOPOLOGY ON THE SPECTRUM AND STRICTLY NORMING FAMILIES

Let us discuss now in more detail the relation between the concept of invertibility sufficient family and the simpler (to check) concept of an exhausting family. The following theorem studies C^* -algebras with the property that every invertibility sufficient family is also exhausting. It explains Example 3.13 and Remark 3.14.

Lemma 4.1. *Let A be a C^* -algebra, J a two-sided ideal, and π a representation of A such that π is nondegenerate on J . Also let $a \in A$, $0 \leq a \leq 1$, such that $\|\pi(a)\| = 1$ and choose $\eta > 0$. Then there exists $c \in J$, $c \geq 0$, $\|c\| \leq \eta$ such that $\|\pi(a + c)\| \geq 1 + \eta/2$.*

Proof. For any fixed $\varepsilon > 0$ there exists a unit vector ξ such that $\langle \pi(a)\xi, \xi \rangle \geq 1 - \varepsilon$. Let us consider then the positive linear form $\varphi : A \rightarrow \mathbb{C}$ defined by $\varphi(b) := \langle \pi(b)\xi, \xi \rangle$. If (u_λ) is an approximate unit in J , then

$$\|\varphi\| \geq \|\varphi|_J\| = \lim \varphi(u_\lambda) = \|\xi\| = 1.$$

So $\|\varphi|_J\| = \|\phi\| = 1$. Hence there exists $c_0 \in J$, $c_0 \geq 0$, $\|c_0\| = 1$, such that $\varphi(c_0) \geq 1 - \varepsilon$. We then set $c = \eta c_0$ and indeed, for ε small enough

$$\|a + c\| \geq \varphi(a + c) \geq 1 - \varepsilon + \eta(1 - \varepsilon) \geq 1 + \eta/2.$$

This completes the proof. \square

We shall use the above lemma in the form of the following corollary.

Corollary 4.2. *Let π_0 be an irreducible representation of a C^* -algebra A and let $I_0 := \ker(\pi_0) \in \text{Prim}(A)$. We assume that we are given decreasing sequence $V_0 \supset \dots \supset V_n \supset V_{n+1} \dots$ of open neighborhoods of I_0 in $\text{Prim}(A)$. Then there exists $a \in I_0$ such that $\|a\| = \|\pi_0(a)\| = 1$ and $\|\pi(a)\| \leq 1 - 2^k$ for any irreducible representation π such that $\ker(\pi) \notin V_k$.*

Proof. To construct $a \in A$ with the desired properties, let us consider the ideals J_n defining the sets V_n , that is, $V_n = \text{Prim}(J_n)$, $n \geq 0$. Since $V_n \subset V_{n-1}$ for all n , we have that $J_n \subset J_{n-1}$ for all n .

The element a we are looking for will be the limit of a sequence (a_n) , $a_n \in A$, where the a_n are defined inductively to satisfy the following properties:

- (i) $0 \leq a_n \leq 1$;
- (ii) $\|\pi_0(a_n)\| = 1$;
- (iii) $\|\pi(a_n)\| \leq 1 - 2^{-k}$ for all irreducible representations π such that $\ker(\pi) \in \text{Prim}(A) \setminus \text{Prim}(J_k)$ for $k = 0, 1, \dots, n$;
- (iv) $\|a_n - a_{n-1}\| \leq 2^{2-n}$ for $n \geq 1$.

We define the initial term a_0 as follows. We first choose $b_0 \in J_0$ such that $0 \leq b_0$, and $\pi_0(b_0) \neq 0$. By rescaling b_0 with a positive factor, we can assume that $\|\pi_0(b_0)\| = 1$. Let then $\chi_0 : [0, \infty) \rightarrow [0, 1]$ be the continuous function defined by $\chi_0(t) = t$ for $t \leq 1$ and $\chi_0(t) = 1$ for $t \geq 1$. Then we define $a_0 = \chi_0(b_0)$. Conditions (i-iv) are then satisfied.

Next, a_n is defined in terms a_{n-1} . In order to do that, we first define auxiliary elements c_n and $b_n = a_{n-1} + c_n$ as follows. By Lemma 4.1, there exists $c_n \in J_n$, $c_n \geq 0$, $\|c_n\| \leq 2^{1-n}$, such that $\|\pi_0(b_n)\| \geq 1 + 2^{-n}$. Let then $\chi_n : [0, \infty) \rightarrow [0, 1]$ be the continuous function defined by $\chi_n(t) = t$ for $t \leq 1 - 2^{1-n}$, χ_n linear on $[1 - 2^{1-n}, 1]$ and on $[1, 1 + 2^{-n}]$, $\chi_n(1) = 1 - 2^{-n}$, and $\chi_n(t) = 1$ for $t \geq 1 + 2^{-n}$. Then we define $a_n = \chi_n(b_n)$.

Claim. The sequence $a_n \in A$ just constructed satisfies conditions (i-iv).

Indeed, we have checked our conditions for $n = 0$, so let us assume $n \geq 1$ and check our conditions for $a_n \in A$ one by one:

- (i) We have that $a_{n-1}, c_n \geq 0$, hence $b_n := a_{n-1} + c_n \geq 0$. Since $0 \leq \chi_n \leq 1$, we obtain that $0 \leq a_n := \chi_n(b_n) \leq 1$.
- (ii) Since $0 \leq \chi_n \leq 1$, $\chi_n(t) = 1$ for $t \geq 1 + 2^{-n}$, and $\|\pi_0(b_n)\| \geq 1 + 2^{-n}$, we have that $\|\pi_0(a_n)\| = \|\pi_0(\chi_n(b_n))\| = \|\chi_n(\pi_0(b_n))\| = 1$.
- (iii) Let $\pi \in \hat{A}$ be such that $\ker(\pi) \in \text{Prim}(J_k)^c := \text{Prim}(A) \setminus \text{Prim}(J_k)$, for some k , $0 \leq k \leq n$. We need to check that $\|\pi(a_n)\| \leq 1 - 2^{-k}$.

We have that π vanishes on J_k , and hence $\pi(c_n) = 0$ since $c_n \in J_n \subset J_k$, $k \leq n$. Therefore,

$$\pi(a_n) = \pi(\chi_n(b_n)) = \chi_n(\pi(b_n)) = \chi_n(\pi(a_{n-1})).$$

We shall consider now two cases: $k < n$ and $k = n$.

Case 1. If $k < n$, then $\|\pi(a_{n-1})\| \leq 1 - 2^{-k} \leq 1 - 2^{1-n}$, by the induction hypothesis. Since $\chi_n(t) = t$ for $t \leq 1 - 2^{1-n}$, we obtain $\pi(a_n) = \chi_n(\pi(a_{n-1})) = \pi(a_{n-1})$, and hence $\|\pi(a_n)\| = \|\pi(a_{n-1})\| \leq 1 - 2^{-k}$ for $k < n$.

Case 2. If $k = n$, we have $\|\pi(a_n)\| = \|\chi_n(\pi(a_{n-1}))\| \leq 1 - 2^{-n} = 1 - 2^{-k}$, since $\pi(a_n) = \chi_n(\pi(a_{n-1}))$, $\chi_n(t) \leq 1 - 2^{-n}$ for $t \leq 1$, and $0 \leq a_{n-1} \leq 1$.

(iv) We have $\|b_n\| \leq \|a_{n-1}\| + \|c_n\| \leq 1 + 2^{1-n}$. Since $|\chi_n(t) - t| \leq 2^{1-n}$ for all $t \leq 2^{1-n}$, we have $\|a_n - b_n\| \leq 2^{1-n}$. Hence

$$\|a_n - a_{n-1}\| \leq \|a_n - b_n\| + \|b_n - a_{n-1}\| \leq 2^{1-n} + \|c_n\| \leq 2^{2-n}.$$

This completes the proof of our claim, and hence the sequence a_n constructed above satisfies Conditions (i-iv).

Let us now show how to use the fact that the sequence $a_n \in A$ satisfies Conditions (i-iv) to construct a as in the statement of this corollary. First of all, Condition (iv) allows us to define $a := \lim_{n \rightarrow \infty} a_n$. Let us show that $a \in A$ satisfied the desired conditions. Since Conditions (i-iii) are compatible with limits, we have

- (i) $0 \leq a \leq 1$;
- (ii) $\|\pi_0(a)\| = 1$;
- (iii) $\|\pi(a)\| \leq 1 - 2^{-k}$ for all irreducible representations π such that $\ker(\pi) \in \text{Prim}(A) \setminus \text{Prim}(J_k)$ for $k \geq 0$.

Thus a has the properties stated in this corollary, which completes the proof. \square

Proposition 4.3. *Let A be a unital C^* -algebra. Let us assume that every $I \in \text{Prim}(A)$ has a countable base for its system of neighborhoods. Then every strictly norming family \mathcal{F} of representations of A is also exhausting.*

Let us assume that $\text{Prim}(A)$ is a T_1 space. Then the converse is also true, that is, if every strictly norming family \mathcal{F} of representations of A is also exhausting, then every $I \in \text{Prim}(A)$ has a countable base for its system of neighborhoods.

We think that the condition that $\text{Prim}(A)$ be T_1 is not necessary. However, as noticed by Roch, the proof below requires this assumption.

Proof. Let us prove first the first part of the statement, so let us assume that every primitive ideal $I \in \text{Prim}(A)$ has a countable base for its system of neighborhoods and let \mathcal{F} be a strictly norming family of representations of A . We need to show that \mathcal{F} is exhausting. We shall proceed by contradiction. Thus, let us assume that the family \mathcal{F} is not exhausting. Then there exists a primitive ideal $I_0 = \ker(\pi_0) \in \text{Prim}(A) \setminus \bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$. Let

$$V_0 \supset \dots \supset V_n \supset V_{n+1} \dots \supset \{I_0\} = \bigcap_k V_k$$

be a basis for the system of neighborhoods of I_0 in $\text{Prim}(A)$. We may assume without loss of generality that the neighborhoods V_n consist of open sets. Corollary 4.2 then yields $a \in A$ such that $\|a\| = \|\pi_0(a)\| = 1$, but $\|\pi(a)\| \leq 1 - 2^{-k}$ for any irreducible representation π of A such that $\ker(\pi) \in \text{Prim}(A) \setminus V_k$. Then, for every $\phi \in \mathcal{F}$, we have that

$$\text{Prim}(A) \setminus \text{supp}(\phi) = \{I \in \text{Prim}(A), \ker(\phi) \not\subset I\} = \text{Prim}(\ker(\phi))$$

is an open subset of $\text{Prim}(A)$ containing I_0 , and hence it is a neighborhood of I_0 in $\text{Prim}(A)$. Therefore there exists n such that $V_n \subset \text{Prim}(A) \setminus \text{supp}(\phi)$ and hence $\|\pi(a)\| \leq 1 - 2^{-n}$ for all π such that $\ker(\pi) \in \text{supp}(\phi)$. This gives $\|\phi(a)\| \leq$

$1 - 2^{-n} < 1$, thus contradicting the fact that \mathcal{F} is strictly norming. This proves the first half of the statement.

Let us prove the converse, that is, the second half of the statement, which is easier. Thus let us assume that every strictly norming family of representations of A is also exhausting and let us prove that every primitive ideal $I_0 := \ker(\pi_0) \in \text{Prim}(A)$ has a countable basis for its system of neighborhoods. Let us fix then $I_0 := \ker(\pi_0) \in \text{Prim}(A)$ arbitrarily and show that it has a countable basis for its system of neighborhoods. Also, we associate to each primitive ideal $I \in \text{Prim}(A)$ an irreducible representation ϕ_I with kernel I . By remark 3.9, we have that the family of representations $\mathcal{F} := \{\phi_I, I \in \text{Prim}(A), I \neq I_0\}$ is not exhausting, since $\text{Prim}(A)$ is a T_1 space (and hence its points are closed) and hence $\text{supp } \phi_I = I$. By our assumption, the family \mathcal{F} is hence also not strictly norming. Therefore, by the definition of a strictly norming family of representations, there exists $a \in A$, such that $\|\pi(a)\| < \|a\|$ for all irreducible π with $\ker(\pi) \neq I_0$. Note that since the family \widehat{A} is strictly norming (see Example 3.2), we have that $\|a\| = \max_{\pi \in \widehat{A}} \|\pi(a)\|$, and hence $\|a\| = \|\pi_0(a)\|$. By rescaling, we can assume $\|a\| = \|\pi_0(a)\| = 1$. Then the sets

$$V_n := \{ \ker(\pi) \in \text{Prim}(A), \|\pi(a)\| > 1 - 2^{-n} \}$$

are open neighborhoods of $I_0 := \ker(\pi_0)$ in $\text{Prim}(A)$ by Lemma 1.2. Let us show that they form a basis for the system of neighborhoods of I_0 . Indeed, let G be an arbitrary open subset of $\text{Prim}(A)$ containing I_0 . Then there exists a two-sided ideal $J \subset A$ such that $G = \text{Prim}(J)$. The set of irreducible representations of A/J identifies with $\text{Prim}(J)^c := \text{Prim}(A) \setminus \text{Prim}(J)$, and hence it does not contain π_0 . Hence $\|\pi(a)\| < 1$ for all $\pi \in \text{Prim}(A/J)$. Since $\widehat{A/J}$ is a strictly norming family of representations of A/J , we obtain that $\|a + J\|_{A/J} < 1$ (the norm is in A/J). Let n be such that $\|a + J\|_{A/J} \leq 1 - 2^{-n}$. Then $V_n \subset \text{Prim}(J) = G$, which completes the proof of the second half of this theorem. The proof is now complete. \square

Clearly, there are C^* -algebras for which the spectrum is not T_1 , but for which every strictly norming family of representations is also exhausting. We do not know, however, if the converse result is true in full generality (that is, for every C^* -algebra). It is easy to show that separable C^* -algebras satisfy the assumptions of Proposition 4.3.

Theorem 4.4. *Let A be a separable C^* -algebra. Then every primitive ideal $I \in \text{Prim}(A)$ has a countable base for its system of neighborhoods. Consequently, if \mathcal{F} is a strictly norming set of representations of A , then \mathcal{F} is exhausting.*

Proof. It is known [14] that $\text{Prim}(A)$ is second countable. This gives the result in view of Proposition 4.3. For the benefit of the reader, we now provide a quick proof that every point in $\text{Prim}(A)$, for A separable, has a countable base for its system of neighborhoods. Indeed, we can replace A with A^+ and thus assume that A is unital. Let $\{a_n\}$ be a dense subset of A and fix $I_0 := \ker(\pi_0) \in \text{Prim}(A)$. Define

$$V_n := \{ \ker(\pi) \in \text{Prim}(A), \|\pi(a_n)\| > \|\pi_0(a_n)\|/2 \}.$$

Then each V_n is open by Lemma 1.2. We claim that V_n is a basis of the system of neighborhoods of $I_0 := \ker(\pi_0)$ in $\text{Prim}(A)$. Indeed, let $G \subset \text{Prim}(A)$ be an open set containing I_0 . Then $G = \text{Prim}(J)$ for some two-sided ideal of A such that $\pi_0 \neq 0$ on J . Let $a \in J$ such that $\pi_0(a) \neq 0$. By the density of the sequence a_n in A ,

we can find n such that $\|a - a_n\| < \|\pi_0(a)\|/4$. Then $\|\pi'(a) - \pi'(a_n)\| < \|\pi_0(a)\|/4$ for any irreducible representation π' , and hence

$$(11) \quad \|\pi'(a)\| - \|\pi_0(a)\|/4 < \|\pi'(a_n)\| < \|\pi'(a)\| + \|\pi_0(a)\|/4, \quad \forall \pi' \in \hat{A}.$$

To show that $V_n \subset G$, it is enough to show that $V_n \cap G^c = V_n \cap \text{Prim}(J)^c = \emptyset$. Suppose the contrary and let $\pi \in \hat{A}$ be such that $\ker(\pi) \in V_n \cap \text{Prim}(J)^c$. Then $\|\pi(a_n)\| > \|\pi_0(a_n)\|/2$, by the definition of V_n . Moreover, $\pi(a) = 0$ since $a \in J$ and π vanishes on J . Let us show that this is not possible. Indeed, using Equation (11) twice, for $\pi' = \pi_0$ and for $\pi' = \pi$, we obtain

$$\frac{3}{8}\|\pi_0(a)\| < \frac{1}{2}\|\pi_0(a_n)\| < \|\pi(a_n)\| < \frac{1}{4}\|\pi_0(a)\|,$$

which is contradiction. Consequently $V_n \subset G$ and hence $\{V_n\}$ is a basis for the system of neighborhoods of π_0 in $\text{Prim}(A)$, as claimed. The last part follows from the first part of Proposition 4.3. \square

The next two basic examples illustrate the differences between the notions of faithful and strictly norming families.

Example 4.5. Let in this example A be the C^* -algebra of continuous functions f on $[0, 1]$ with values in $M_2(\mathbb{C})$ such that $f(1)$ is diagonal, which is a type I C^* -algebra, and thus we identify \hat{A} and $\text{Prim}(A)$. Then the maps $\text{ev}_t: f \mapsto f(t) \in M_2(\mathbb{C})$, for $t < 1$, together with the maps $\text{ev}_1^i: f \mapsto f(1)_{ii}$ ($i = 0, 1$) provide all the irreducible representations of A (up to equivalence). The family

$$\mathcal{F} = \{\text{ev}_t, t < 1\} \cup \{\text{ev}_1^1\}$$

is a faithful but not exhausting family. In fact the function $t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1-t \end{pmatrix}$ is not invertible in A but $\pi(f)$ is invertible for all $\pi \in \mathcal{F}$. Of course, in this example, every $\pi \in \hat{A} = \text{Prim}(A)$ has a countable base for its system of neighborhoods, so every strictly norming family of representations \mathcal{F} of A is also exhausting.

The next example is closely related to the examples we will be dealing with below.

Example 4.6. Let \mathcal{T} be the Toeplitz algebra, which is again a type I C^* -algebra, and thus we again identify $\hat{\mathcal{T}}$ and $\text{Prim}(\mathcal{T})$. The Toeplitz algebra \mathcal{T} is defined as the C^* -algebra generated by the operator defined by the unilateral shift S . (Recall that S acts on the Hilbert space $L^2(\mathbb{N})$ by $S: \epsilon_k \mapsto \epsilon_{k+1}$.) As $S^*S = 1$ and $SS^* - 1$ is a rank 1 operator, one can prove that the following is an exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T} \rightarrow \mathcal{C}(S^1) \rightarrow 0,$$

where \mathcal{K} is the algebra of compact operator. Extend the unique irreducible representation π of \mathcal{K} to \mathcal{T} as in [14]. Also, the irreducible characters χ_θ of S^1 pull-back to irreducible characters of \mathcal{T} vanishing on \mathcal{K} . Then the spectrum of \mathcal{T} is

$$\hat{\mathcal{T}} = \{\pi\} \cup \{\chi_\theta, \theta \in S^1\},$$

with S^1 embedded as a closed subset. A subset $V \subset \text{Prim}(\mathcal{T})$ will be open if, and only if, it contains π and its intersection with S^1 is open. We thus see that the single element set $\{\pi\}$ defines an exhausting family. In other words $\hat{\mathcal{T}} = \overline{\{\pi\}} = \text{supp}(\pi)$. Since every exhausting family is also strictly norming, by Proposition 3.12, the family $\{\pi\}$ consisting of a single representation is also strictly norming. We can see

also directly that the family $\mathcal{F} = \{\pi\}$ (consisting of π alone) is strictly norming. Indeed, it suffices to notice that $\|x\| = \|\pi(x)\|$ for all x since π is injective. In this example again every $\pi' \in \hat{\mathcal{T}} = \text{Prim}(\mathcal{T})$ has a countable base for its system of neighborhoods, so every strictly norming family of representations \mathcal{F} of \mathcal{T} is also exhausting.

Here are two more examples that show that the condition that A be separable is not necessary for the classes of exhausting families of representations and strictly norming families of representations to coincide.

Example 4.7. Let I be an infinite *uncountable* set. We endow it with the discrete topology. Then $A_0 := C_0(I)$ and $A_1 := \mathcal{K}(\ell^2(I))$ (the algebra of compact operators on $\ell^2(I)$) are not separable, however, if \mathcal{F} is a strictly norming family of representations of A_i , $i = 0, 1$, then \mathcal{F} is also an exhausting family of representations of A_i .

5. UNBOUNDED OPERATORS

The results of the previous sections are relevant often in applications to unbounded operators, so we now extend Theorem 3.4 to (possibly) unbounded operators affiliated to C^* -algebras. We begin with an abstract setting.

5.1. Abstract affiliated operators. The notion of affiliated self-adjoint operator has been extensively and successfully studied, see [5, 9, 18, 46, 47] for example. In the sequel we will closely follow the definitions in [18], beginning with an abstract version of this notion. See [21, 35] for results on unbounded operators on Hilbert modules [8, 22, 28].

Definition 5.1. Let A be a C^* -algebra. An *observable* T *affiliated to* A is a morphism $\theta_T : C_0(\mathbb{R}) \rightarrow A$ of C^* -algebras. The observable T is said to be *strictly affiliated to* A if the space generated by elements of the form $\theta_T(h)a$ ($a \in A$, $h \in C_0(\mathbb{R})$), is dense in A .

As in the classical case, we now introduce the Cayley transform. To this end, let us notice that an observable affiliated to A extends to a morphism $\theta_T^+ : C_0(\mathbb{R})^+ \rightarrow A^+$ (the algebra obtained from A by adjunction of a unit). If moreover T is strictly affiliated to A , then θ_T extends to a morphism from $C_b(\mathbb{R})$ to the multiplier algebra of A , but we shall not need this fact.

Definition 5.2. Let T be an observable affiliated to A . The *Cayley transform* $u_T \in A^+$ of T is

$$(12) \quad u_T := \theta_T^+(h_0), \quad h_0(z) := (z + \iota)(z - \iota)^{-1}.$$

The Cayley transform allows us to reduce questions about the spectrum of an observable to questions about the spectrum of its Cayley transform. Let us first introduce, however, the *spectrum of an affiliated observable*. Let thus $\theta_T : C_0(\mathbb{R}) \rightarrow A$ be a self-adjoint operator affiliated to a C^* -algebra A . The kernel of θ_T is then of the form $C_0(U)$, for some open subset of \mathbb{R} . We define the spectrum $\text{Spec}_A(T)$ as the complement of U in \mathbb{R} . Explicitly,

$$(13) \quad \text{Spec}_A(T) = \{\lambda \in \mathbb{R}, h(\lambda) = 0, \forall h \in C_0(\mathbb{R}) \text{ such that } \theta_T(h) = 0\}.$$

We allow the case $\text{Spec}_A(T) = \emptyset$, which corresponds to the case $T = \infty$ or $u_T = 1$. If $\sigma : A \rightarrow B$ is a morphism of C^* -algebras, then $\sigma \circ \theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow A$ is an observable $\sigma(T)$ affiliated to the C^* -algebra B and

$$(14) \quad \text{Spec}_B(\sigma(T)) \subset \text{Spec}_A(T) .$$

If σ is injective, then $\text{Spec}_B(\sigma(T)) = \text{Spec}_A(T)$, which shows that the spectrum is preserved by increasing the C^* -algebra A . Note that

$$(15) \quad \sigma(u_T) = u_{\sigma(T)} .$$

By classical results, if $(u_T - 1)$ is injective, then we can define a true self-adjoint operator $T := \iota(u_T + 1)(u_T - 1)^{-1} \in A$ such that $\theta_T(h) = h(T)$, $h \in \mathcal{C}_0(\mathbb{R})$ [13]. This is the case, for instance, if $\text{Spec}(T)$ is a bounded subset of \mathbb{R} , in which case we shall say that T is *bounded*. In any case, bounded or unbounded, our definition of $\text{Spec}(T)$ in terms of θ_T coincides with the classical spectrum of T defined using the resolvent. Let $h_0(z) := (z + \iota)(z - \iota)^{-1}$, as before.

Lemma 5.3. *The spectrum $\text{Spec}(T)$ of the an observable $\theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow A$ affiliated to the C^* -algebra A and the spectrum $\text{Spec}(u_T)$ of its Cayley transform are related by*

$$\text{Spec}(T) = h_0^{-1}(\text{Spec}(u_T)) .$$

Proof. This follows from the fact that h_0 is a homeomorphism of \mathbb{R} onto its image in $S^1 := \{|z| = 1\}$ and from the properties of the functional calculus. \square

Let us notice that the above lemma is valid also in the case when

$$T = \infty \Leftrightarrow \theta_T = 0 \Leftrightarrow \text{Spec}(T) = \emptyset \Leftrightarrow u_T = 1 \Leftrightarrow \sigma(u_T) = \{1\} .$$

One can make the relation in the above lemma more precise by saying that, for bounded T , we have $h_0(\text{Spec}(T)) = \text{Spec}(u_T)$, whereas for unbounded T we have

$$(16) \quad \overline{h_0(\text{Spec}(T))} = h_0(\text{Spec}(T)) \cup \{1\} = \text{Spec}(u_T) ,$$

where $h_0(z) := (z + \iota)(z - \iota)^{-1}$, as before.

Here is our main result on (possibly unbounded) self-adjoint operators affiliated to C^* -algebras.

Theorem 5.4. *Let A be a unital C^* -algebra and T an observable affiliated to A . Let \mathcal{F} be a set of representations of A .*

(1) *If \mathcal{F} is strictly norming, then*

$$\text{Spec}(T) = \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(T)) .$$

(2) *If \mathcal{F} is faithful, then*

$$\text{Spec}(T) = \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(T))} .$$

Proof. The proofs of (i) and (ii) are similar, starting with the relation $\text{Spec}(T) = h_0^{-1}(\text{Spec}(u_T))$ of Lemma 5.3. We begin with (i), which is slightly easier. Since \mathcal{F} is strictly norming, we can then apply theorem 3.6 to $u_T \in A^+$ and the family $\sigma \in \mathcal{F}$. We obtain

$$\begin{aligned} \text{Spec}(T) &= h_0^{-1}[\text{Spec}(u_T)] = h_0^{-1}[\cup_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(u_T))] \\ &= h_0^{-1}[\cup_{\sigma \in \mathcal{F}} \text{Spec}(u_{\sigma(T)})] = \cup_{\sigma \in \mathcal{F}} h_0^{-1}[\text{Spec}(u_{\sigma(T)})] = \cup_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(T)) . \end{aligned}$$

If, on the other hand, \mathcal{F} is faithful, we apply proposition 2.8 after noting that h_0 is a homeomorphism of \mathbb{R} onto its image in $S^1 := \{|z| = 1\}$ and hence $h_0^{-1}(\overline{S}) = \overline{h_0^{-1}(S)}$ for any $S \subset S^1$. The same argument then gives

$$\begin{aligned} \text{Spec}(T) &= h_0^{-1}[\text{Spec}(u_T)] = h_0^{-1}[\bigcup_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(u_T))] \\ &= h_0^{-1}[\bigcup_{\sigma \in \mathcal{F}} \text{Spec}(u_{\sigma(T)})] = \bigcup_{\sigma \in \mathcal{F}} h_0^{-1}[\text{Spec}(u_{\sigma(T)})] = \bigcup_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(T)). \end{aligned}$$

The proof is now complete. \square

Remark 5.5. In view of the remarks preceding it, Theorem 5.4 remains valid for true self-adjoint operators T .

5.2. The case of ‘true’ operators. We now look at concrete (true) operators.

Definition 5.6. Let $A \subset \mathcal{L}(\mathcal{H})$ be a sub- C^* -algebra of $\mathcal{L}(\mathcal{H})$. A (possibly unbounded) self-adjoint operator $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ is said to be *affiliated* to A if, for every continuous functions h on the spectrum of T vanishing at infinity, we have $h(T) \in A$.

Remark 5.7. We have that T is affiliated to A if, and only if, $(T - \lambda)^{-1} \in A$ for one $\lambda \notin \text{Spec}(T)$ (equivalently for all such λ) [9]. We thus see that a self-adjoint operator T affiliated to A defines a morphism $\theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow A$, $\theta_T(h) := h(T)$ such that $\text{Spec}(T) = \text{Spec}(\theta_T)$. Thus T defines an observable affiliated to A .

Since in our paper we shall consider only the case when $A \subset \mathcal{L}(\mathcal{H})$ is non degenerate, we shall not make a difference between operators and observables affiliated to A . Recall that an unbounded operator T is invertible if, and only if, it is bijective and T^{-1} is bounded. This is also equivalent to $0 \notin \text{Spec}(\theta_T)$. We have the following analog of Proposition 2.5 and Theorem 3.4

Theorem 5.8. *Let $A \subset \mathcal{L}(\mathcal{H})$ be a unital C^* -algebra and T a self-adjoint operator affiliated to A . Let \mathcal{F} be a set of representations of A .*

- (1) *Let \mathcal{F} be strictly norming. Then T is invertible if, and only if $\phi(T)$ is invertible for all $\phi \in \mathcal{F}$.*
- (2) *Let \mathcal{F} be faithful. Then T is invertible if, and only if $\phi(T)$ is invertible for all $\phi \in \mathcal{F}$ and the set $\{\|\phi(T)^{-1}\|, \phi \in \mathcal{F}\}$ is bounded.*

Proof. This follows from Theorem 5.4 as follows. First of all, we have that T is invertible if, and only if, $0 \notin \text{Spec}(T)$. Now, if \mathcal{F} is strictly norming, we have

$$0 \notin \text{Spec}(T) \Leftrightarrow 0 \notin \bigcup_{\phi \in \mathcal{F}} \text{Spec}(\phi(T)) \Leftrightarrow 0 \notin \text{Spec}(\phi(T)) \text{ for all } \phi \in \mathcal{F}.$$

This proves (i). (Note that $\phi(T)$ may not be a true operator, but only and affiliated observable.) To prove (ii), we proceed similarly, noticing also that the distance from 0 to $\text{Spec}(T)$ is exactly $\|T^{-1}\|$. \square

We have already remarked (Remark 5.5) that Theorem 5.4 extends to the framework of this subsection, that is, that of (possibly unbounded) self-adjoint operators on a Hilbert space.

6. PARAMETRIC PSEUDODIFFERENTIAL OPERATORS

Let M be a compact smooth Riemannian manifold and G be a Lie group (finite dimensional) with Lie algebra $\mathfrak{g} := \text{Lie}(G)$. We let G act by left translations on $M \times G$. We denote by $\Psi^0(M \times G)^G$ the algebra of order 0, G -invariant pseudodifferential operators on $M \times G$ and $\overline{\Psi^0(M \times G)^G}$ be its norm closure acting on $L^2(M \times G)$. For any vector bundle E , we denote by S^*E the set of directions in its dual E^* . If E is endowed with a metric, then S^*E can be identified with the set of unit vectors in E^* . We shall be interested in the quotient

$$S^*(T(M \times G))/G = S^*(TM \times TG)/G = S^*(TM \times \mathfrak{g}).$$

We have that $\overline{\Psi^0(M \times G)^G} \simeq C_r^*(G) \otimes \mathcal{K}$ and then obtain the exact sequence

$$(17) \quad 0 \rightarrow C_r^*(G) \otimes \mathcal{K} \rightarrow \overline{\Psi^0(M \times G)^G} \rightarrow \mathcal{C}(S^*(M \times \mathfrak{g})) \rightarrow 0,$$

[24, 25, 29, 46]. Note that the kernel of the symbol map will now have irreducible representations parametrized by \hat{G}_r , the temperate unitary irreducible representations of G . Let $T \in \Psi^m(M \times G)^G$ and denote by $T^\sharp \in \Psi^m(M \times G)^G$ its formal adjoint (defined using the calculus of pseudodifferential operators). All operators considered below are closed with minimal domain (the closure of the operators defined on $\mathcal{C}_c^\infty(M \times G)$). We denote by T^* the Hilbert space adjoint of a (possibly unbounded) densely defined operator.

Lemma 6.1. *Let $T \in \Psi^m(M \times G)^G$ be elliptic. Then $T^* = T^\sharp$. Thus, if also $T = T^\sharp$, then T is self-adjoint and $(T + \iota)^{-1} \in C_r^*(G)$, and hence it is affiliated to $C_r^*(G)$.*

Proof. This is a consequence of the fact that $\Psi^\infty(M \times G)^G$ is closed under multiplication and formal adjoints. See [24, 25, 46] for details. \square

In other words, any elliptic, formally self-adjoint $T \in \Psi^m(M \times G)^G$, $m > 0$, is actually self-adjoint.

Let us assume $G = \mathbb{R}^n$, regarded as an abelian Lie group. Then our exact sequence (17) becomes

$$(18) \quad 0 \rightarrow \mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K} \rightarrow \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}} \rightarrow \mathcal{C}(S^*(TM \times \mathbb{R}^n)) \rightarrow 0.$$

This shows that $A := \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$ is a type I C^* -algebra, and hence we can identify \hat{A} and $\text{Prim}(A)$. Then we use that, to each $\lambda \in \mathbb{R}^n$, there corresponds an irreducible representation ϕ_λ of $\mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K}$. Recalling that every irreducible (bounded, *) representation of an ideal I in a C^* -algebra A extends uniquely to a representation of A , we obtain that ϕ_λ extends uniquely to an irreducible representation of $\overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$ denoted with the same letter. It is customary to denote by $\hat{T}(\lambda) := \phi_\lambda(T)$ for T a pseudodifferential operator in $\Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$, $m \geq 0$. To define $\hat{T}(\lambda)$ for $m > 0$, we can either use the Fourier transform or, notice that Δ is affiliated to the closure of $\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}$. This allows us to define $\hat{\Delta}(\lambda)$. In general, we write $T = (1 - \Delta)^k S$, with $S \in \Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}$ and define $\hat{T}(\lambda)q = (\widehat{1 - \Delta})(\lambda)^k \hat{S}(\lambda)$. (We consider the “analyst’s” Laplacian, so $\Delta \leq 0$.)

Lemma 6.2. *Let $A := \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$. Then the primitive ideal spectrum of A , $\text{Prim}(A)$, is in a canonical bijection with the disjoint union $\mathbb{R}^n \cup S^*(TM \times \mathbb{R}^n)$, where the copy of \mathbb{R}^n corresponds to the open subset $\{\phi_\lambda, \lambda \in \mathbb{R}^n\}$ and the copy of $S^*(TM \times \mathbb{R}^n)$ corresponds to the closed subset $\{e_p, p \in S^*(TM \times \mathbb{R}^n)\}$. The*

induced topologies on \mathbb{R}^n and $S^*(TM \times \mathbb{R}^n)$ are the standard ones. Let $S^*M := S^*(TM) \subset S^*(TM \times \mathbb{R}^n)$ correspond to $T^*M \subset T^*M \times \mathbb{R}^n$. Then the closure of $\{\phi_\lambda\}$ in $\text{Prim}(A)$ is $\{\phi_\lambda\} \cup S^*M$.

Proof. By standard properties of C^* -algebras (the definition of the Jacobson topology), the ideal $\mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K} \subset A$ defines an open subset of $\text{Prim}(A)$ with complement $\text{Prim}(A/I)$ with the induced topologies. This proves the first part of the statement.

In order to determine the closure of $\{\phi_\lambda\}$ in $\text{Prim}(A)$, let us notice that the principal symbol of $\hat{T}(\lambda)$ can be calculated in local coordinate charts on M (more precisely, on sets of the form $U \times \mathbb{R}^n$, with U a coordinate chart in M). This gives that the principal symbol of $\hat{T}(\lambda)$ is given by the restriction of the principal symbol of T to S^*M .

Indeed, let $U = \mathbb{R}^k$. A translation invariant pseudodifferential P operator on $U \times \mathbb{R}^n = \mathbb{R}^{k+n}$ is of the form $P = a(x, y, D_x, D_y)$ with a independent of y : $a(x, y, \xi, \eta) = \tilde{a}(x, \xi, \eta)$. With this notation, we have $\hat{P}(\lambda) = \tilde{a}(x, D_x, \lambda)$. The principal symbol of $\hat{P}(\lambda)$ is then the principal symbols of the (global) symbol $\mathbb{R}^k \ni (x, \xi) \rightarrow \tilde{a}(x, \xi, \lambda)$, and is seen to be independent of the (finite) value of $\lambda \in \mathbb{R}^n$ and is the restriction from $S^*(TU \times \mathbb{R}^n)$ to $S^*(TU \times \{0\})$ of the principal symbol of \tilde{a} .

Returning to the general case, the same reasoning gives that the image of ϕ_λ is $\overline{\Psi^0(M)}$. The primitive ideal spectrum of this algebra is canonically homeomorphic to the closure of $\{\phi_\lambda\}$, and this is enough to complete the proof. \square

By the exact sequence (17), in addition to the irreducible representations ϕ_λ , $\lambda \in \mathbb{R}^n$ (or, more precisely, their kernels), $\text{Prim}(A)$ contains also (the kernels of) the irreducible representations $e_p(T) = \sigma_0(T)(p)$, $p \in S^*(TM \times \mathbb{R}^n)$.

Proposition 6.3. *Let $\mathcal{F} := \{\phi_\lambda, \lambda \in \mathbb{R}^n\} \cup \{e_p, p \in S^*(TM \times \mathbb{R}^n) \setminus S^*M\}$.*

- (i) *The family \mathcal{F} is a strictly norming family of representations of $\overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$.*
- (ii) *Let $P \in \Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$, then $P : H^s(M \times \mathbb{R}^n) \rightarrow H^{s-m}(M \times \mathbb{R}^n)$ is invertible if, and only if $\hat{P}(\lambda) : H^s(M) \rightarrow H^{s-m}(M)$ is invertible for all $\lambda \in \mathbb{R}^n$ and the principal symbol of P is non-zero on all rays not intersecting S^*M .*
- (iii) *If $T \in \Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$, $m > 0$, is formally self-adjoint and elliptic, then $\text{Spec}(e_p(T)) = \emptyset$, and hence*

$$\text{Spec}(T) = \cup_{\lambda \in \mathbb{R}^n} \text{Spec}(\hat{T}(\lambda)).$$

Proof. (i) follows from Lemma 6.2. To prove (ii), let us denote by $\Delta_M \leq 0$ the (non-positive) Laplace operator on M . Then the Laplace operator Δ on $M \times \mathbb{R}^n$ is $\Delta = \Delta_{\mathbb{R}^n} + \Delta_M$. Note that $(1 - \Delta)^{m/2} : H^s(M \times \mathbb{R}^n) \rightarrow H^{s-m}(M \times \mathbb{R}^n)$ and $(c - \Delta_M)^{m/2} : H^s(M) \rightarrow H^{s-m}(M)$, $c > 0$, are isomorphisms. By [25], we have that

$$P_1 := (1 - \Delta)^{(s-m)/2} P (1 - \Delta)^{-s/2} \in A := \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}.$$

It is then enough to prove that P_1 is invertible on $L^2(M \times \mathbb{R}^n)$. Moreover from part (i) we have just proved and Theorem 3.4 we know that P_1 is invertible on $L^2(M \times \mathbb{R}^n)$ if, and only if, $\hat{P}_1(\lambda) := \phi_\lambda(P_1)$ is invertible on $L^2(M)$ for all $\lambda \in \mathbb{R}^n$ and the principal symbol of P_1 is non-zero on all rays not intersecting S^*M . But, using also $\widehat{1 - \Delta}(\lambda) = (1 + |\lambda|^2 - \Delta_M)$, we have

$$\hat{P}_1(\lambda) = (1 + |\lambda|^2 - \Delta_M)^{(s-m)/2} \hat{P}(\lambda) (1 + |\lambda|^2 - \Delta_M)^{-s/2},$$

which is invertible by assumption.

To prove (iii), we recall that T is affiliated to A , by Lemma 6.1. The result then follows from Theorem 5.4(1) (See also Remark 5.5). \square

Operators of the kind considered in this subsection were used also in [1, 7, 10, 26, 30, 43, 44]. They turn out to be useful also for general topological index theorems [16, 34]. A more class of operators than the ones considered in this subsection were introduced in [3, 4]. The above result has turned out to be useful for the study of layer potentials [36]. There are, of course, many other relevant examples, but developing them would require too much additional materials, so we plan to discuss these other examples somewhere else.

REFERENCES

- [1] B. Ammann, R. Lauter, and V. Nistor. Pseudodifferential operators on manifolds with a Lie structure at infinity. *Ann. of Math. (2)*, 165(3):717–747, 2007.
- [2] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*, volume 36 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. L'Enseignement Mathématique, Geneva, 2000. With a foreword by G. Skandalis and Appendix B by E. Germain.
- [3] I. Androulidakis and G. Skandalis. The holonomy groupoid of a singular foliation. *J. Reine Angew. Math.*, 626:1–37, 2009.
- [4] I. Androulidakis and G. Skandalis. Pseudodifferential calculus on a singular foliation. *J. Noncommut. Geom.*, 5(1):125–152, 2011.
- [5] Saad Baaj and Pierre Julg. Théorie bivariante de Kasparov et opérateurs non bornés dans les C^* -modules hilbertiens. *C.R. Acad. Sci. Paris*, 296(21):875–878, 1983.
- [6] M. Buneci. Groupoid C^* -algebras. *Surv. Math. Appl.*, 1:71–98 (electronic), 2006.
- [7] A. Connes. Sur la théorie non commutative de l'intégration. In *Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978)*, volume 725 of *Lecture Notes in Math.*, pages 19–143. Springer, Berlin, 1979.
- [8] A. Connes. *Noncommutative geometry*. Academic Press, San Diego, 1994.
- [9] M. Damak and V. Georgescu. Self-adjoint operators affiliated to C^* -algebras. *Rev. Math. Phys.*, 16(2):257–280, 2004.
- [10] C. Debord and J.-M. Lescure. K -duality for stratified pseudomanifolds. *Geom. Topol.*, 13(1):49–86, 2009.
- [11] C. Debord, J.-M. Lescure, and F. Rochon. Pseudodifferential operators on manifolds with fibred corners. preprint arXiv:1112.4575, to appear in *Annales de l'Institut Fourier*.
- [12] C. Debord and G. Skandalis. Adiabatic groupoid, crossed product by \mathbb{R}_+^* and pseudodifferential calculus. *Adv. Math.*, 257:66–91, 2014.
- [13] J. Dereziński and C. Gérard. *Scattering theory of classical and quantum N -particle systems*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.
- [14] J. Dixmier. *Les C^* -algèbres et leurs représentations*. Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Paris, 1996. Reprint of the second (1969) edition.
- [15] S. Echterhoff and D. Williams. Inducing primitive ideals. *Trans. Amer. Math. Soc.*, 360(11):6113–6129, 2008.
- [16] H. Emerson and R. Meyer. Equivariant embedding theorems and topological index maps. *Adv. Math.*, 225(5):2840–2882, 2010.
- [17] R. Exel. Invertibility in groupoid C^* -algebras. In *Operator theory, operator algebras and applications*, volume 242 of *Oper. Theory Adv. Appl.*, pages 173–183. 2014.
- [18] V. Georgescu and Andrei Iftimovici. C^* -algebras of quantum Hamiltonians. In *Operator algebras and mathematical physics (Constanța, 2001)*, pages 123–167. Theta, Bucharest, 2003.
- [19] V. Georgescu and V. Nistor. On the essential spectrum of N -body hamiltonians with asymptotically homogeneous interactions. <http://arxiv.org/abs/1506.03267> [math.SP].
- [20] M. Ionescu and D. Williams. The generalized Effros-Hahn conjecture for groupoids. *Indiana Univ. Math. J.*, 58(6):2489–2508, 2009.

- [21] J. Kaad and M. Lesch. A local global principle for regular operators in Hilbert C^* -modules. *J. Funct. Anal.*, 262(10):4540–4569, 2012.
- [22] E. C. Lance. *Hilbert C^* -modules*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995.
- [23] Y. Last and B. Simon. The essential spectrum of Schrödinger, Jacobi, and CMV operators. *J. Anal. Math.*, 98:183–220, 2006.
- [24] R. Lauter, B. Monthubert, and V. Nistor. Pseudodifferential analysis on continuous family groupoids. *Doc. Math.*, 5:625–655 (electronic), 2000.
- [25] R. Lauter and V. Nistor. Analysis of geometric operators on open manifolds: a groupoid approach. In *Quantization of singular symplectic quotients*, volume 198 of *Progr. Math.*, pages 181–229. Birkhäuser, Basel, 2001.
- [26] M. Lesch and M. Pflaum. Traces on algebras of parameter dependent pseudodifferential operators and the eta-invariant. *Trans. Amer. Math. Soc.*, 352(11):4911–4936, 2000.
- [27] M. Măntoiu. C^* -algebras, dynamical systems at infinity and the essential spectrum of generalized Schrödinger operators. *J. Reine Angew. Math.*, 550:211–229, 2002.
- [28] V. M. Manuilov and E. V. Troitsky. *Hilbert C^* -modules*, volume 226 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2005. Translated from the 2001 Russian original by the authors.
- [29] B. Monthubert. Pseudodifferential calculus on manifolds with corners and groupoids. *Proc. Amer. Math. Soc.*, 127(10):2871–2881, 1999.
- [30] B. Monthubert and F. Pierrot. Indice analytique et groupoïdes de Lie. *C.R. Acad. Sci. Paris*, 325(2):193–198, 1997.
- [31] S. A. Nazarov and B. A. Plamenevsky. *Elliptic problems in domains with piecewise smooth boundaries*, volume 13 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.
- [32] V. Nistor. Analysis on singular spaces: Lie manifolds and operator algebras. <http://arxiv.org/abs/1512.06575> [math.OA].
- [33] V. Nistor. Desingularization of Lie groupoids and pseudodifferential operators on singular spaces. <http://arxiv.org/abs/1512.08613> [math.DG].
- [34] V. Nistor and Evgenij Troitsky. The Thom isomorphism in gauge-equivariant K -theory. In *C^* -algebras and elliptic theory*, Trends Math., pages 213–245. Birkhäuser, Basel, 2006.
- [35] A. Pal. Regular operators on Hilbert C^* -modules. *J. Operator Theory*, 42(2):331–350, 1999.
- [36] Yu Qiao and V. Nistor. Single and double layer potentials on domains with conical points I: Straight cones. *Integral Equations Operator Theory*, 72(3):419–448, 2012.
- [37] V. Rabinovich, S. Roch, and B. Silberman. *Limit operators and their applications in operator theory*, volume 150 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2004.
- [38] J. Renault. *A groupoid approach to C^* -algebras*, volume 793 of *LNM*. Springer, 1980.
- [39] J. Renault. The ideal structure of groupoid crossed product C^* -algebras. *J. Operator Theory*, 25(1):3–36, 1991. With an appendix by G. Skandalis.
- [40] M. Rieffel. Induced representations of C^* -algebras. *Advances in Math.*, 13:176–257, 1974.
- [41] S. Roch. Algebras of approximation sequences: structure of fractal algebras. In *Singular integral operators, factorization and applications*, volume 142 of *Oper. Theory Adv. Appl.*, pages 287–310. Birkhäuser, Basel, 2003.
- [42] S. Roch, P. Santos, and B. Silberman. *Non-commutative Gelfand theories*. Universitext. Springer-Verlag London, Ltd., London, 2011.
- [43] E. Schrohe. Fréchet algebra techniques for boundary value problems on noncompact manifolds: Fredholm criteria and functional calculus via spectral invariance. *Math. Nachr.*, 199:145–185, 1999.
- [44] E. Schrohe and B.-W. Schulze. Boundary value problems in Boutet de Monvel’s algebra for manifolds with conical singularities. I. In *Pseudo-differential calculus and mathematical physics*, volume 5 of *Math. Top.*, pages 97–209. Akademie Verlag, Berlin, 1994.
- [45] B.-W. Schulze. *Pseudo-differential operators on manifolds with singularities*, volume 24 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., 1991.
- [46] Stéphane Vassout. Unbounded pseudodifferential calculus on Lie groupoids. *J. Funct. Anal.*, 236(1):161–200, 2006.
- [47] S. L. Woronowicz. C^* -algebras generated by unbounded elements. *Rev. Math. Phys.*, 7(3):481–521, 1995.

UNIVERSITÉ DE LORRAINE, UFR MIM, ILE DU SAULCY, CS 50128, 57045 METZ, FRANCE
E-mail address: `victor.nistor@univ-lorraine.fr`

UNIVERSITÉ DE LORRAINE, UFR MIM, ILE DU SAULCY, CS 50128 57045 METZ, FRANCE
E-mail address: `nicolas.prudhon@univ-lorraine.fr`